Review: Continuity - 10/12/16

1 Continuity Definition

Definition 1.0.1 *A function $f(x)$ is* ***continuous at*** *a if $\lim_{x\to a} f(x) = f(a)$.*

Notice that there are three things that we need to check when evaluating this definition:

1. a is in the domain of f. If it's not, then $f(a)$ is not defined, so we can't have that the limit equals $f(a)$.
2. $\lim_{x\to a} f(x)$ exists. We can't have the limit equal $f(a)$ if the limit doesn't even exist. For example, see the jump discontinuity: $f(a)$ exists, but the limit doesn't.
3. Given that both exist, finally we check that $\lim_{x\to a} f(x) = f(a)$.

Example 1.0.2 *Is $f(x) = \frac{x^2-x-2}{x+1}$ continuous at -1. First, is -1 in the domain of f? It's not, so we can stop right here - since -1 is not in the domain of f, then f cannot be continuous there.*

Example 1.0.3 *Is*

$$h(x) = \begin{cases} \frac{x^2-x-2}{x+1} & x \neq -1 \\ -3 & x = -1 \end{cases}$$

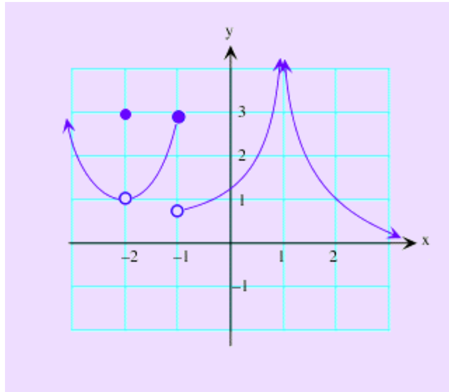
continuous at -1? First, is -1 in the domain of h? It is - $h(-1) = -3$. Next, does the limit exist? We can rewrite g as $\frac{(x-2)(x+1)}{x+1}$, so to find the limit, we cancel out the $x+1$s to get that $\lim_{x\to -1} g(x) = \lim_{x\to -1} x - 2 = -3$. Finally, is the limit equal to $h(-1)$? Yes! Thus $h(x)$ is continuous at -1.

If $f(x)$ is not continuous at a, but it is defined for all x around a (except perhaps at a), then we say that $f(x)$ is **discontinuous** at a. So for example, $f(x) = \frac{x^2-x-2}{x+1}$ is discontinuous at -1, but $g(x) = \sqrt{x}$ is not discontinuous at -2 because it's not defined for anything near it.

Definition 1.0.4 *A function is* ***continuous on an interval*** *if it is continuous at every point in that interval.*

2 Types of Discontinuities

Example 2.0.5 *Which points on the graph are discontinuities (points at which $f(x)$ is discontinuous)?*



*$x = -2$ is a **removable discontinuity** because if we just redefined that point, we could make the graph continuous. $x = -1$ is a **jump discontinuity** because the function jumps up to a new point. $x = 1$ is an **infinite discontinuity** because it goes off to infinity.*

3 Properties of Continuity

If f and g are continuous at a, and if c is a function, then the following functions are continuous:

- $f + g$
- $f - g$
- cf
- fg
- $\frac{f}{g}$ if $g(a) \neq 0$

Rule for composition limits:

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) \text{ if } f \text{ is continuous at } \lim_{x \to a} g(x).$$

As an extension of this rule, we have **if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.**

4 Types of Continuous Functions

Remember that functions are continuous if they don't have any holes, jumps, or asymptotes. What kinds of functions are continuous?

The following types functions are continuous **on their domains**:

- Polynomial
- Rational
- Roots
- Trigonometric
- Logarithmic
- Exponential

5 Examples

Example 5.0.6 *Is*

$$f(x) = \begin{cases} x^2 - 2 & x < 1 \\ -\sqrt{x} & x \geq 1 \end{cases}$$

a continuous function? Each of the pieces are continuous functions, so we just need to check if it is continuous at the point where they switch, namely at 1. Is 1 in the domain? Yes, $f(1) = -\sqrt{1} = -1$. *Does the limit exist? Here, we have to check the left and right hand limits to see if they match. As we come from the left, we are coming from the part of the function that is* $x^2 - 2$*, so we have that* $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 - 2 = 1^2 - 2 = -1$*. From the right, we are approaching 1 using the* $-\sqrt{x}$ *part, so we have* $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -\sqrt{x} = -\sqrt{1} = -1$*. Since the right and left limits are the same, we have that the overall limit exists and equals -1. Since that's the same as our value for* $f(1)$*, this function is continuous.*

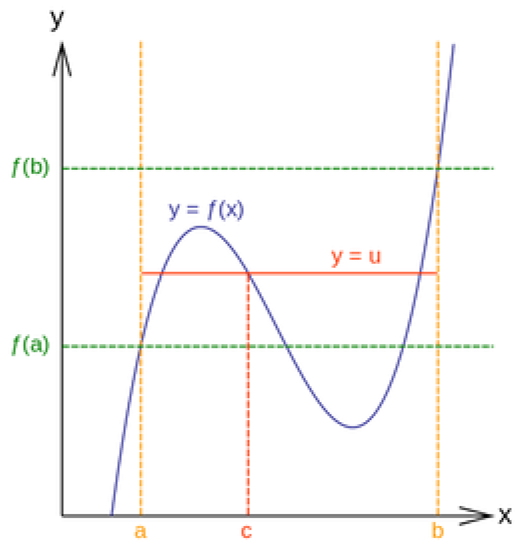
Example 5.0.7 *Let* $g(x) = \frac{x^2-2x}{x^2+x-6}$*. Where is* g *discontinuous? What kind of discontinuities are they? Because rational functions are continuous on their domains, we just have to check what points aren't in the domain - those points will be the discontinuities. If we factor the top and bottom, we get* $g(x) = \frac{x(x-2)}{(x-2)(x+3)}$*. Thus the points not in the domain are* 2 *and* -3*, so these are the points where* g *is discontinuous. Since we can remove the problem caused by 2 by canceling* $(x-2)$ *top and bottom, 2 is a removable discontinuity (it is a hole). We can't remove -3, so this is an infinite discontinuity (it is an asymptote).*

Example 5.0.8 *Is* $h(x) = \cos\left(\frac{x^2+1}{x-1}\right)$ *continuous at 0? Notice that if we let* $f(x) = \cos(x)$ *and* $g(x) = \frac{x^2+1}{x-2}$*, then* $h(x) = (f \circ g)(x)$*. Our rule of composition says that if* $g(x)$ *is continuous at* a *and* $f(x)$ *is continuous at* $g(a)$*, then* $f(g(x))$ *is continuous at* a*. Here,* $a = 0$*, so is* $g(x)$ *continuous at 0? Yes,* $g(x)$ *is a rational function, so it is continuous on its domain, and 0 is in its domain. Is* $f(x)$ *continuous at* $g(0)$*? Well,* cos *is continuous everywhere. Thus* $h(x)$ *is continuous at 0.*

6 Intermediate Value Theorem

Theorem 6.0.9 ***Intermediate Value Theorem*** *Suppose that* f *is continuous on the closed interval* $[a, b]$ *and let* N *be any number between* $f(a)$ *and* $f(b)$*. Then there exists a number* c*, with* $a < c < b$ *so that* $f(c) = N$*.*

Below is a picture illustrating the theorem. In the picture, N is labeled as $y = u$.



Example 6.0.10 *Use the Intermediate Value Theorem to show that $f(x) = x^3 - 6x + 2$ has a zero between 0 and 1. To start with, we know that polynomials are continuous everywhere, so $f(x)$ is continuous on the closed interval $[0, 1]$. Now we can use the intermediate value theorem! We're trying to find a zero of the function, so we're trying to show that there is a place c between 0 and 1 where $f(c) = 0$. Thus we just need to show that $N = 0$ is in between $f(0)$ and $f(1)$. Notice that $f(0) = 2$, and $f(1) = -3$, and zero is in between those. So since f is continuous on the interval $[0, 1]$, and 0 is between $f(0)$ and $f(1)$, then the IVT says there exists a number c between 0 and 1 so that $f(c) = 0$. In other words, $f(x)$ has a zero between 0 and 1.*

Practice Problems

1. Is

$$f(x) = \begin{cases} \frac{x^2-3x+2}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$$

continuous? If not, where is it discontinuous? What kind of discontinuity is it? Can you make a new piecewise function with a different value for $x = 2$ so that the function is now continuous?

2. Is $\sin\left(\frac{1}{x}\right)$ continuous at $x = 3$? Is it continuous at $x = 0$?

3. Use the IVT to show that $g(x) = x^5 + x - 1$ has a zero between 0 and 1.

4. Let $h(x) = \frac{x^2-4}{x^2-3x+2}$. On what intervals is $h(x)$ continuous? Where are its discontinuities? What kind of discontinuities are they?

5. Let

$$f(x) = \begin{cases} x^3 + \pi x - 1 & x < 0 \\ x + 5 & x \geq 0 \end{cases}.$$

On what intervals is $f(x)$ continuous? Where are its discontinuities? What kind of discontinuities are they?

Solutions

1. It is not continuous at $x = 2$. This is a removable discontinuity. A new function would have $f(x) = 1$ when $x = 2$.

2. It is continuous at $x = 3$ because 3 is in the domain of $\frac{1}{x}$ and sin is a continuous function, so we have the composition of two continuous functions. It is not continuous at $x = 0$ because 0 is not in the domain.

3. Since $g(x) = x^5 + x - 1$ is a polynomial, it is continuous everywhere, so it is continuous on the closed interval $[0, 1]$. We have $g(0) = -1$ and $g(1) = 1$, so since 0 is between those two, the intermediate value theorem says there exists a number c between 0 and 1 so that $g(c) = 0$.

4. $h(x)$ is continuous everywhere on its domain, which is $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$. It has discontinuities at $x = 1$ and $x = 2$. $x = 2$ is a removable discontinuity because $(x - 2)$ cancels with an $x - 2$ in the numerator. $x = 1$ is an infinite discontinuity because there is no way to cancel it (it is an asymptote).

5. Since both pieces are polynomials, $f(x)$ is continuous everywhere except possibly zero (where we switch polynomials). To see if it is continuous there, we have to check that the limit exists. But $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^3 + \pi x - 1 = -1$, and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + 5 = 5$. Since the left and right hand limits do not equal each other, the limit does not exist, and $f(x)$ is discontinuous at zero. Thus it is continuous on $(-\infty, 0) \cup (0, \infty)$. Since we jump from -1 to 5 at $x = 0$, this is a jump discontinuity.